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Quantum symmetries associated with the Perk–Schultz model

M Couture

AECL, Chalk River Laboratories, Chalk River, Ontario, Canada K0J 1J0

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Abstract. The quantum algebraic structures associated with a family of R-matrices one extracts from the Perk–Schultz model are studied. Starting with the quantum spaces one can define from the R-matrices, we show how different duality conditions at the level of quantum spaces (Manin's construction) translate in different quantum groups and quantized universal enveloping algebras.

1. Introduction

In this paper we study the quantum algebraic structures associated with the R-matrices $\check{R}$ one extracts from the Perk–Schultz vertex model [1]. Throughout the paper $\check{R}$ denotes the matrices which are solutions of

$$\check{R}_{12}\check{R}_{23}\check{R}_{12}=\check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{1.1}$$

where the subscripts indicate the action on the triple tensor product space $V\otimes V\otimes V$ where V is an n-dimensional complex vector space. The exact solution of this model has been recently discussed in [2]. Recent studies (reviewed below) suggest that in addition to the usual supersymmetry there exist another symmetry associated with this model. It is the purpose of this paper to examine this question. In Manin's construction of quantum groups [3–5] one considers a set of quantum spaces on which the quantum group coacts. We shall consider the coaction on a pair of quantum spaces which are dual to each other. These quadratic algebras are defined using the matrices $\check{R}$ one extracts (by a limiting procedure) from the model. Two cases are considered each corresponding to a particular definition of duality (at the level of the quantum spaces) and rule for the multiplication in the tensor product of two algebras. At the level of the quantized universal enveloping algebras this translates into different duality conditions expressed in terms of the solutions of the graded or non-graded Yang–Baxter equations (without spectral parameter). Examination of the two-dimensional case suggests that these two constructions lead to two versions of supersymmetry, one of which does not have a classical limit; in this simple case, the version of supersymmetry which does not have a classical limit is shown to be related to $U_t(\mathrm{sl}(2,\boldsymbol{C}))$ at root of unity ($t=i$, $i^2=-1$).

All algebras discussed are Z_2 graded and we shall use some of Manin's notation [5] in describing the grading. The Z_2-degree of an element b will be denoted $\hat{b}$; a format is an arbitrary sequence $(a_1, a_2, \ldots, a_n)$ with $a_i\in Z_2$. Every algebra will be associated with a given format by defining the grading (parity) $\hat{z}_i^j$ of its elements z_i^j

$(i,j=1,n)$ as $\hat{z}_i^j=a_i+a_j$; we put $\hat{i}=a_i$ and therefore $\hat{z}_i^j=\hat{i}+\hat{j}$. The solutions of (1.1) one extracts from these graded vertex models can now be written as follows:

$$\check{R}(q)=(1-q^2)\sum_{l\leq i<j\leq n} e_i^i\otimes e_j^j+\sum_{i=1}^{n}(-1)^{\hat{i}}q^{2\hat{i}}e_i^i\otimes e_i^i+q\sum_{\substack{i\neq j\\ i,j=1}}^{n}(-1)^{\alpha(\hat{i},\hat{j})}e_i^j\otimes e_j^i \tag{1.2}$$

where e_i^j is a matrix unit: $(e_i^j)_m^n=\delta_i^m\delta_j^n$, α is an arbitrary function of $\hat{i}$ and $\hat{j}$, the only restriction being that $\alpha(\hat{i},\hat{j})=\alpha(\hat{j},\hat{i})\in Z_2$. Due to the simplicity of $\check{R}$, it is easily verified that it satisfies (1.1) for all $\alpha(\hat{i},\hat{j})$. For a given format $(a_1,a_2,\ldots,a_n)$, the characteristic and minimal polynomials $Ch(\lambda)$ and $m(\lambda)$ of $\check{R}$ are as follows:

$$Ch(\lambda)=(\lambda-1)^{[n(n+1)/2]-\Sigma a_i}(\lambda+q^2)^{[n(n-1)/2]+\Sigma a_i} \tag{1.3a}$$

$$m(\lambda)=(\lambda-1)(\lambda+q^2). \tag{1.3b}$$

Note that given (1.2) one can obtain the statistical weights of this model through Baxterization using the following formula

$$W(X;q)=\check{R}(q)+\lambda_1\lambda_2X\check{R}^{-}(q) \tag{1.4}$$

where from (1.3) we have $\lambda_1=1$ and $\lambda_2=-q^2$. With the identification $q=e^{i\nu}$ and $X=e^{2i\theta}$ it follows that (trigonometric regime)

$$W(\theta;\nu)=\sum_{\substack{a\neq b\\ a,b=1}}^{n} e^{i\theta\,\mathrm{sign}(a-b)}e_a^a\otimes e_b^b+\sum_{a=1}^{n}\sin(\nu+(-1)^{\hat{a}}\theta)\sin(\nu)^{-1}e_a^a\otimes e_a^a$$
$$+\sum_{\substack{a\neq b\\ a,b=1}}^{n}(-1)^{\alpha(\hat{a},\hat{b})}\sin(\theta)\sin(\nu)^{-1}e_b^a\otimes e_a^b. \tag{1.5}$$

These weights correspond to those of de Vega and Lopes [2] where their function G_{ab} is equal to $(-1)^{\alpha(\hat{a},\hat{b})}$.

The quantum algebraic structures associated to the family of solutions (1.2) have already been the object of several studies. It is well known that $\mathrm{GL}_q(1|1)$ is related to the solution $n=2$ with format $(0,1)$ which we denote $\check{R}(0,1)$; certain aspects of it have been discussed in [6]; in [7] the differential geometry and quantized universal enveloping algebra of $\mathrm{GL}_q(1|1)$ was presented. In [8], the quantum Lie superalgebra $\mathrm{sl}_q(M|N)$ was shown to be related to these solutions. In [9], a supersymmetric version of the R-formalism of the St. Petersburg school [10] was discussed and, starting with $\check{R}(0,1)$, the quantized universal enveloping algebra related to $\mathrm{GL}_q(1|1)$ was derived as an example. All of these results are examples of quantum deformations of Lie superalgebras and of the group $\mathrm{GL}(1|1)$.

Other quantum algebraic structures have been shown to be related to the solutions (1.2). Starting with $\check{R}(0,1)$, Jing *et al* [11] used the standard (non-supersymmetric) version of the R-formalism (the distinction between the two versions of the R-formalism will be made clear in the following sections) and obtained a quantized universal enveloping algebra different from the one obtained in [7, 9]. In [12], it was shown that $\check{R}(0,1)$ is related to the two dimensional highest weight representation of $\mathrm{U}_t(\mathrm{sl}(2,\boldsymbol{C}))$ with $t=i$; the parameter q that appears in $\check{R}(0,1)$ is no longer the deformation parameter (since $t=i$) but it is the free parameter that characterizes the representation (in general the representations of $\mathrm{U}_t(\mathrm{sl}(2,\boldsymbol{C}))$ at roots of unity are parameterized by three parameters). In the case $q=i$, the two parameter quantized universal enveloping algebra $\mathrm{U}_{q,s}(\mathrm{gl}(K+1;\boldsymbol{C});L)$ presented in [13] is also related to the solutions (1.2).

In all of the above studies, the discussion was restricted to the quantized universal enveloping algebras associated with the solutions (1.2) (except in the case of $\mathrm{GL}_q(1|1)$). The approach used in this paper differs from the ones discussed above in that we begin our analysis at the level of quantum spaces for which quantum groups are the symmetry transformations. The quantized universal enveloping algebras are objects which are then defined in a space dual to that of the quantum group. The symmetries involved are then more transparent. Our paper is therefore organized as follows. In section 2 the quantum spaces are defined. The homomorphisms of two sets S and $\bar{S}$ are examined in sections 3 and 4. We then proceed to define the quantized universal enveloping algebras in sections 5 and 6. In section 7 the particular case of $\check{R}(0, 1)$ is discussed in detail. We conclude with a few remarks. Since we intend to draw a parallel between two possible paths there will necessarily be some overlap with previous works; these will be indicated as we proceed.

2. Associated quantum spaces

The quantum spaces associated to the solutions (1.2) are quadratic algebras generated by n variables $x_1, x_2, \ldots, x_n$ subject to the relations

$$[f(\check{R})]^{kl}_{ij} x_k x_l = 0 \tag{2.1}$$

where f is an arbitrary polynomial in $\check{R}$; it follows from (1.3b) that f can always be written

$$f(\check{R}) = c_1 \check{R} + c_2 I \tag{2.2}$$

where I is the unit matrix and c_1, $c_2 \in C$ are to be determined by solving (2.1). There are only two solutions that lead to non-zero x_i's. The first solution $c_1 = -c_2$ leads to

$$\begin{aligned} &x_i^2 = 0 \qquad \hat{i} = 1 \\ &x_i x_j - (-1)^{\alpha(\hat{i},\hat{j})} q^{-1} x_j x_i = 0 \qquad i<j \end{aligned} \tag{2.3}$$

while from the second solution $c_2 = c_1 q^2$ we get

$$\begin{aligned} &x_i^2 = 0 \qquad \hat{i} = 0 \\ &x_i x_j + (-1)^{\alpha(\hat{i},\hat{j})} q x_j x_i = 0 \qquad i<j. \end{aligned} \tag{2.4}$$

We first connect with the one parameter version of Manin's [5] general linear supergroup $\mathrm{GL}_q(M|N)$. For ease of reference we use most of his notation.

3. $\mathrm{GL}_q(M|N)$

Manin introduces the following two quantum spaces (we consider Manin's one parametric version). A_q is a quadratic algebra generated by n coordinates $x_1, \ldots, x_n$ with parity assignment $\hat{x}_i = \hat{i}$ and commutation rules

$$\begin{aligned} &(x_i)^2 = 0 \qquad \text{for} \quad \hat{i} = 1 \\ &x_i x_j - (-1)^{\hat{i}\hat{j}} q^{-1} x_j x_i = 0 \qquad \text{for} \quad i<j. \end{aligned} \tag{3.1}$$

Note that the relations defining A_q corresponds to the solution (2.3) with $\alpha(\hat{i},\hat{j})=\hat{i}\hat{j}$. The quadratic algebra A_q^* is generated by n coordinates $\xi^1,\ldots,\xi^n$ with parity assignment $\hat{\xi}^k=1+\hat{k}$ and commutation rules

$$\begin{aligned}&(\xi^k)^2=0 \qquad \text{for} \quad \hat{k}=0\\ &\xi^k\xi^l-(-1)^{(\hat{k}+1)(\hat{l}+1)}q\xi^l\xi^k=0 \qquad \text{for} \quad k<l.\end{aligned} \tag{3.2}$$

He defines duality through the following pairings

$$\langle\xi^j;x_i\rangle=\delta_i^j \tag{3.3a}$$

$$\langle\xi^k\otimes\xi^l;x_i\otimes x_j\rangle=(-1)^{\hat{i}(\hat{l}+1)}\delta_i^k\delta_j^l. \tag{3.3b}$$

One may check that following this definition A_q^* is dual to A_q. The rule for multiplication in the tensor product of two algebras is defined to be

$$(a\otimes b)(c\otimes d)=(-1)^{\hat{b}\hat{c}}(ac\otimes bd). \tag{3.4}$$

Now consider the $n\times n$ matrix $Z=(z_i^j)_{i,j=1}^n\in \mathrm{GL}_q(M|N)$ associated to a format $(a_1,a_2,\ldots,a_n)$ where $M=n-\Sigma_{i=1}^n a_i$ and $N=\Sigma_{i=1}^n a_i$. The commutation relations satisfied by the elements z_i^j are determined by requiring that the maps

$$\begin{aligned}&\delta(x)=Z\otimes x\Rightarrow\delta(x_i)=\sum_{j=1}^n z_i^j\otimes x_j\\ &\delta^*(\xi)=Z\otimes\xi\Rightarrow\delta(\xi^k)=\sum_{l=1}^n z_k^l\otimes\xi^l\end{aligned} \tag{3.5}$$

be homomorphism of A_q and A_q^* respectively; with the rule (3.4) one obtains Manin's relations which we give for ease of reference

$$\begin{aligned}&(z_i^k)^2=0 \qquad \hat{i}+\hat{k}=\text{odd}\\ &z_i^kz_i^l-(-1)^{(\hat{k}+1)(\hat{l}+1)}qz_i^lz_i^k=0 \qquad \hat{i}=\text{odd} \qquad k<l\\ &z_i^kz_i^l-(-1)^{\hat{k}\hat{l}}q^{-1}z_i^lz_i^k=0 \qquad \hat{i}=\text{even} \qquad k<l\\ &z_i^kz_j^k-(-1)^{\hat{i}\hat{j}}q^{-1}z_j^kz_i^k=0 \qquad \hat{k}=\text{even} \qquad i<j\\ &z_i^kz_j^k-(-1)^{(\hat{i}+1)(\hat{j}+1)}qz_j^kz_i^k=0 \qquad \hat{k}=\text{odd} \qquad i<j\\ &(-1)^{\hat{k}(\hat{j}+\hat{l})}z_i^kz_j^l-(-1)^{\hat{i}(\hat{j}+\hat{l})}z_j^lz_i^k=(-1)^{\hat{j}\hat{l}}(q^{-1}-q)z_i^lz_j^k \qquad i<j,\,k<l\\ &z_j^kz_i^l=(-1)^{(\hat{j}+\hat{k})(\hat{i}+\hat{l})}z_i^lz_j^k \qquad i<j,\,k<l.\end{aligned} \tag{3.6}$$

The last relation in (3.6) follows by requiring that $q^2\neq-1$.

We stress that the map (coproduct)

$$\Delta(z_i^k)=\sum_{j=1}^n z_i^j\otimes z_j^k \tag{3.7}$$

preserves the structure described in (3.6) provided one uses the rule (3.4). We now use the graded permutation operator $\tilde{P}$ and define a matrix $\tilde{R}$

$$\tilde{R}=\tilde{P}\check{R} \qquad [\tilde{P}]_{ab}^{cd}=\delta_a^d\delta_b^c(-1)^{\hat{c}\hat{d}} \tag{3.8a}$$

$$\tilde{R}=q\sum_{\substack{i\neq j\\ i,j=1}}^n e_i^i\otimes e_j^j+\sum_{i=1}^n q^{2\hat{i}}e_i^i\otimes e_i^i+(1-q^2)\sum_{1\leq i<j\leq n}(-1)^{\hat{i}\hat{j}}e_j^i\otimes e_i^j. \tag{3.8b}$$

The matrix elements $\tilde{R}^{\gamma\delta}_{\phi\beta}$ satisfy the graded Yang-Baxter equations ($\tilde{R}^{\gamma\delta}_{\phi\beta}$ is equal to the coefficient of $e^{\gamma}_{\phi}\otimes e^{\delta}_{\beta}$ in (3.8b))

$$\sum_{\phi,\beta,\gamma=1}^{n} \tilde{R}^{\phi\beta}_{ij}\tilde{R}^{r\gamma}_{\phi k}\tilde{R}^{sp}_{\beta\gamma}(-1)^{\hat{i}\hat{j}+\hat{\phi}\hat{k}+\hat{\beta}\hat{\gamma}} = \sum_{\phi,\beta,\gamma=1}^{n} \tilde{R}^{\phi\beta}_{jk}\tilde{R}^{\gamma p}_{i\beta}\tilde{R}^{rs}_{\gamma\phi}(-1)^{\hat{j}\hat{k}+\hat{i}\hat{\beta}+\hat{\gamma}\hat{\phi}} \tag{3.9a}$$

which can be written

$$\tilde{R}_{12}\eta_{12}\tilde{R}_{13}\eta_{13}\tilde{R}_{23}\eta_{23} = \tilde{R}_{23}\eta_{23}\tilde{R}_{13}\eta_{13}\tilde{R}_{12}\eta_{12}. \tag{3.9b}$$

The use of the matrices $(\eta_{ij})^{b_1b_2b_3}_{a_1a_2a_3} = \delta^{b_1}_{a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}(-1)^{\hat{b}_i\hat{b}_j}$ provide a means of taking into account the gradings while preserving normal matrix multiplication; this way of writing the graded Yang-Baxter equation was introduced in [9]. The relations (3.6) can now be written as follows:

$$\tilde{\tilde{R}}Z_1\tilde{\tilde{P}}Z_1\tilde{\tilde{P}} = \tilde{\tilde{P}}Z_1\tilde{\tilde{P}}Z_1\tilde{\tilde{R}} \qquad Z_1 = Z\otimes I. \tag{3.10}$$

The fundamental representation ρ of $\mathrm{GL}_q(M|N)$ is as follows:

$$\rho(z^j_i)^{\beta}_{\phi} = \tilde{R}^{j\beta}_{i\phi} \tag{3.11}$$

indeed substitution of (3.11) into (3.10), and use of the fact that

$$\tilde{R}^{j\beta}_{i\phi} \neq 0 \Rightarrow i+\phi = j+\beta \qquad \text{and} \qquad \hat{i}\hat{\phi} = \hat{j}\hat{\beta} \tag{3.12}$$

leads to the graded Yang-Baxter equation (3.9). Let $\hat{A} = \boldsymbol{C}\langle z^j_i\rangle$ be a $\boldsymbol{C}$-algebra freely generated by the n^2 variables z^j_i; the algebra of quantum matrices is defined as follows:

$$\tilde{A}_{\tilde{R}} = \boldsymbol{C}\langle z^j_i\rangle / I_{\tilde{R}} \tag{3.13}$$

where $I_{\tilde{R}}$ is the two-sided ideal in $\tilde{A}$ generated by the relations (3.10). Note that if we put $q=1$ in (3.6), the z^j_i's obey the supercommutation rules. $\tilde{A}_{\tilde{R}}$ is therefore a deformation of the ring of polynomial functions on a supermanifold. A few words on related works.

The relation between $\tilde{R}$ in the $n=2$ case with format (0, 1) and $\mathrm{GL}_q(1|1)$ has been discussed in [7]. In [9] the relation (3.10) is given without any connection with a particular quantum group and only a few special cases of the solutions (1.2) are mentioned; what we have shown here is that starting with Manin's quantum planes one is led naturally to this relation. The connection between $\tilde{R}$ as defined in (3.8b) and Manin's relations (3.6) is believed to be new. Note that the above construction remains true for general $\alpha(\hat{i}, \hat{j})$.

We now repeat this construction using a different duality condition and we shall keep α general. Due to its similarities with Manin's quantum group we denote this structure $\overline{\mathrm{GL}}_q((M|N);\alpha)$.

4. $\overline{\mathrm{GL}}_q((M|N);\alpha)$

We denote by $\bar{A}_q$ the quadratic algebra generated by n coordinates $x_1, x_2, \ldots, x_n$ with parity assignment $\hat{x}_i = \hat{i}$ and commutation rules

$$\begin{aligned} &x_i^2 = 0 && \text{for} \quad \hat{i} = 1 \\ &x_ix_j - (-1)^{\alpha(\hat{i},\hat{j})}q^{-1}x_jx_i = 0 && \text{for} \quad i<j \end{aligned} \tag{4.1}$$

where $\alpha(\hat{i},\hat{j})$ is arbitrary; the relation (4.1) correspond to (2.3). Next consider the quadratic algebra $\bar{A}_q^*$ generated by n coordinates $\xi^1, \xi^2, \ldots, \xi^n$ with parity assignment $\hat{\xi}^k = 1+\hat{k}$ and commutation rules

$$
\begin{aligned}
&(\xi^k)^2=0 \qquad \text{for} \quad \hat{k}=0\\
&\xi^k\xi^l+(-1)^{\alpha(\hat{k},\hat{l})}q\xi^l\xi^k=0 \qquad \text{for} \quad k<l
\end{aligned}
\tag{4.2}
$$

(4.2) correspond to solutions (2.4). We note that since relations (4.1) and (4.2) were obtained by solving (2.1) it follows that no additional relations are needed in order to assure associativity of $\bar{A}_q$ and $\bar{A}_q^*$. Defining duality through the following pairings,

$$\langle \xi^j; x_i\rangle = \delta_i^j \tag{4.3a}$$

$$\langle \xi^k\otimes\xi^l|x_i\otimes x_j\rangle = \delta_i^k\delta_j^l \tag{4.3b}$$

it follows that $\bar{A}_q^*$ is dual to $\bar{A}_q$. The multiplication rule between tensor products of two algebras is defined to be

$$(a\otimes b)(c\otimes d)=(ac\otimes bd). \tag{4.4}$$

Let us proceed as in the case of $\mathrm{GL}_q(M|N)$ and consider a matrix $Z=(z_i^j)_{i,j=1}^n \in \overline{\mathrm{GL}}_q((M|N);\alpha)$ with a fixed format and require that the maps (3.5) be homomorphisms of $\bar{A}_q$ and $\bar{A}_q^*$; we obtain the following relations:

$$
\begin{aligned}
&(z_i^k)^2=0 \qquad \hat{i}+\hat{k}=\text{odd}\\
&z_i^kz_i^l+(-1)^{\alpha(\hat{k},\hat{l})}qz_i^lz_i^k=0 \qquad \hat{i}=\text{odd} \qquad k<l\\
&z_i^kz_i^l-(-1)^{\alpha(\hat{k},\hat{l})}q^{-1}z_i^lz_i^k=0 \qquad \hat{i}=\text{even} \qquad k<l\\
&z_i^kz_j^k-(-1)^{\alpha(\hat{i},\hat{j})}q^{-1}z_j^kz_i^k=0 \qquad \hat{k}=\text{even} \qquad i<j\\
&z_i^kz_j^k+(-1)^{\alpha(\hat{i},\hat{j})}qz_j^kz_i^k=0 \qquad \hat{k}=\text{odd} \qquad i<j\\
&z_i^lz_j^k=(-1)^{\alpha(\hat{k},\hat{l})+\alpha(\hat{i},\hat{j})}z_j^kz_i^l \qquad \text{for} \quad i<j,\ k<l\\
&(-1)^{\alpha(\hat{k},\hat{l})}z_i^kz_j^l-(-1)^{\alpha(\hat{i},\hat{j})}z_j^lz_i^k=(q^{-1}-q)z_i^lz_j^k \qquad \text{for} \quad i<j,\ k<l.
\end{aligned}
\tag{4.5}
$$

We now use the non-graded permutation operator P and introduce the family of matrices R

$$
\begin{aligned}
&R=P\check{R} \qquad [P]_{ab}^{cd}=\delta_a^d\delta_b^c\\
&R=q\sum_{\substack{i\neq j\\ i,j=1}}^{n}(-1)^{\alpha(\hat{i},\hat{j})}e_i^i\otimes e_j^j+\sum_{i=1}^{n}(-1)^{\hat{i}}q^{2\hat{i}}e_i^i\otimes e_i^i+(1-q^2)\sum_{1\leq i<j\leq n}e_j^i\otimes e_i^j
\end{aligned}
\tag{4.6}
$$

R is a solution of the non-graded Yang–Baxter equation

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{4.7}$$

The relations (4.5) can now be summarized as follows:

$$RZ_1PZ_1P=PZ_1PZ_1R \qquad Z_1=Z\otimes I. \tag{4.8}$$

The algebra generated by the n^2 elements z_i^j has the following bi-algebra structure

$$\Delta(Z)=Z\otimes Z \Rightarrow \Delta(z_i^k)=\sum_{j=1}^{n} z_i^j\otimes z_j^k$$

$$\varepsilon(Z)=I \Rightarrow \varepsilon(z_i^k)=\delta_i^k.$$

We stress that the difference in the structures described in (3.6) and (4.5) does not lie in the fact that α is kept general (this could have been done in (3.6)) but in the different product rule.

Note that Δ preserves the structure provided one uses the multiplication rule (4.4). The fundamental representation of $\overline{\mathrm{GL}}_q((M|N);\alpha)$ is as follows:

$$\rho(z_i^j)_\phi^\beta = R_{i\phi}^{j\beta} \tag{4.9}$$

indeed substitution of (4.9) into (4.8) leads to (4.7). Let $A = \boldsymbol{C}(z_i^j)$ be the $\boldsymbol{C}$-algebra freely generated by the n^2 variables z_i^j; the algebra of quantum matrices is defined as follows:

$$A_R = \boldsymbol{C}\langle z_i^j\rangle / I_R \tag{4.10}$$

where I_R is the two-sided ideal in A generated by the relations (4.8).

Let us summarize the results obtained thus far. We have considered two sets $S = (A_q, A_q^*)$ and $\bar{S} = (\bar{A}_q, \bar{A}_q^*)$ of quantum spaces associated to the family of solutions $\check{R}$ described in (1.2); note that $A_q = \bar{A}_q$ when $\alpha(\hat{i}, \hat{i}) = \hat{i}\hat{j}$ and in such a case the two sets of planes differ only in the way duality is defined. Examination of the homomorphisms of such spaces has led us in the case of the set S to Manin's $\mathrm{GL}_q(M|N)$ while the set $\bar{S}$ gave us $\overline{\mathrm{GL}}_q((M|N);\alpha)$. We have established that $\overline{\mathrm{GL}}_q((M|N);\alpha)$ is a bialgebra with the multiplication rule (4.4) instead of (3.4) and we suspect that it has a quantum group structure, i.e. that an antipode exists. The determinant also needs to be defined. We shall leave such questions for further studies. We note that the relations defining $\overline{\mathrm{GL}}_q((1|1);\alpha=0)$ with format (0, 1) have also been given in [11]. Finally, it is important to mention that the possibility of having more than one symmetry associated with a given solution of the Yang–Baxter equation (1.1) has already been pointed out by Manin (see example in section 4 of ref [3]) in the context of Yang–Baxter operators which he defines [4] as an operator that satisfies (1.1) but for which $(\check{R})^2 = 1$. These different symmetries are associated to *different Yang–Baxter categories*. $\overline{\mathrm{GL}}_q((M|N);\alpha)$ is an example of this in the case of what Manin refers to as weak Yang–Baxter operators [4], which are operators that satisfy (1.1) but for which $(\check{R})^2 \neq 1$. We now turn to universal enveloping algebras $\tilde{\mathrm{U}}_q$ and U_q associated to $\mathrm{GL}_q(M|N)$ and $\overline{\mathrm{GL}}_q((M|N);\alpha)$ respectively. A study of the dual spaces will allow us to gain a better understanding of the difference between these two symmetries. In particular it will be shown that the quantized universal enveloping algebra of $\overline{\mathrm{GL}}_q((1|1);\alpha=0)$ is related to $\mathrm{U}_{t=i}(\mathrm{sl}(2, \boldsymbol{C}))$.

5. Quantized universal enveloping algebra $\tilde{\mathrm{U}}_q$ associated to $\mathrm{GL}_q(M|N)$

$\tilde{\mathrm{U}}_q$ is defined as a subalgebra of the dual to $\tilde{A}_{\tilde{R}}$; $\tilde{\mathrm{U}}_q$ is generated by the unit element $\mathbf{1}'$ and the generators $L_{ij}^{(\pm)}(i, j = 1, n)$ which are defined by the following duality relations

$$\begin{aligned} \langle 1'; Z_1 Z_2 \ldots Z_k\rangle &= I^{\otimes k} \\ \langle L^{(\pm)}; Z_1 Z_2 \ldots Z_k\rangle &= \tilde{R}_1^{(\pm)} \tilde{R}_2^{(\pm)} \ldots \tilde{R}_k^{(\pm)} \end{aligned} \tag{5.1}$$

where $L^{(\pm)} = (L_{ij}^{(\pm)})_{i,j=1}^n$ and the unit matrix are both $n \times n$ matrices.

$$Z_i = I \otimes \ldots \otimes Z \otimes \ldots \otimes I$$

(k tensor products) and $\tilde{R}_i^{(\pm)}$ act non-trivially on factor number 0 and i in the tensor product $V^{\otimes(k+1)}$ and coincide there with the matrix $\tilde{R}^{(\pm)}$ defined as follows:

$$\tilde{R}^{(+)} = \tilde{P}\tilde{R}\tilde{P} \qquad \tilde{R}^{(-)} = \tilde{R}^{-1} \tag{5.2}$$

where $\tilde{R}$ is defined in (3.8).

In checking the consistency of definition (5.1) with the relation (3.10) use has been made of the properties (3.12) of $\tilde{R}$ and of the fact that $\tilde{R}$ is a solution of (3.9) and of

$$\tilde{R}_{23}\eta_{23}\tilde{R}_{12}^{(-)}\eta_{12}\tilde{R}_{13}^{(-)}\eta_{13} = \tilde{R}_{13}^{(-)}\eta_{13}\tilde{R}_{12}^{(-)}\eta_{12}\tilde{R}_{23}\eta_{23}. \tag{5.3}$$

Due to the particular form and properties of $\tilde{R}$ it follows from (5.1) that $L^{(+)}$ and $L^{(-)}$ are upper and lower triangular matrices respectively and that

$$L_{ii}^{(+)}L_{ii}^{(-)} = 1' \qquad (i = 1, n) \tag{5.4}$$

with no summation over repeated indices. From duality condition (5.1) it can be shown that the following relations exists among the generators of $\tilde{U}_q$

$$\tilde{R}\tilde{P}(L^{(\pm)}\otimes I)\tilde{P}(L^{(\pm)}\otimes I) = (L^{(\pm)}\otimes I)\tilde{P}(L^{(\pm)}\otimes I)\tilde{P}\tilde{R} \tag{5.5a}$$

$$\tilde{R}\tilde{P}(L^{(+)}\otimes I)\tilde{P}(L^{(-)}\otimes I) = (L^{(-)}\otimes I)\tilde{P}(L^{(+)}\otimes I)\tilde{P}\tilde{R}. \tag{5.5b}$$

In verifying the consistency of the relations (5.5) with the defining condition (5.1) we used the property (3.12) of $\tilde{R}$ as well as the fact that $\tilde{R}$ is a solution of (3.9) and

$$\tilde{R}_{13}\eta_{13}\tilde{R}_{23}\eta_{23}\tilde{R}_{12}^{(-)}\eta_{12} = \tilde{R}_{12}^{(-)}\eta_{12}\tilde{R}_{23}\eta_{23}\tilde{R}_{13}\eta_{13} \tag{5.6a}$$

$$\tilde{R}_{12}\eta_{12}\tilde{R}_{23}^{(-)}\eta_{23}\tilde{R}_{13}^{(-)}\eta_{13} = \tilde{R}_{13}^{(-)}\eta_{13}\tilde{R}_{23}^{(-)}\eta_{23}\tilde{R}_{12}\eta_{12}. \tag{5.6b}$$

Writing (5.5) explicitly we get

$$\begin{aligned}
&[L_{ii}^{(+)}, L_{ii}^{(-)}] = 0 && \text{for all } i \\
&(L_{ik}^{(\pm)})^2 = 0 && \hat{i}+\hat{k} = \text{odd} \\
&L_{ik}^{(\pm)}L_{jk}^{(\pm)} - (-1)^{\hat{i}\hat{j}}qL_{jk}^{(\pm)}L_{ik}^{(\pm)} = 0 && \hat{k} = \text{even} \quad i<j \\
&L_{ik}^{(\pm)}L_{jk}^{(\pm)} - q^{-1}(-1)^{(\hat{i}+1)(\hat{j}+1)}L_{jk}^{(\pm)}L_{ik}^{(\pm)} = 0 && \hat{k} = \text{odd} \quad i<j \\
&L_{ik}^{(\pm)}L_{il}^{(\pm)} - q(-1)^{\hat{k}\hat{l}}L_{il}^{(\pm)}L_{ik}^{(\pm)} = 0 && \hat{i} = \text{even} \quad k<l \\
&L_{ik}^{(\pm)}L_{il}^{(\pm)} - q^{-1}(-1)^{(\hat{k}+1)(\hat{l}+1)}L_{il}^{(\pm)}L_{ik}^{(\pm)} = 0 && \hat{i} = \text{odd} \quad k<l \\
&L_{ik}^{(+)}L_{jk}^{(-)} - (-1)^{\hat{i}\hat{j}}q^{-1}L_{jk}^{(-)}L_{ik}^{(+)} = 0 && \hat{k} = \text{even} \quad i<j \\
&L_{ik}^{(+)}L_{jk}^{(-)} - q(-1)^{(\hat{i}+1)(\hat{j}+1)}L_{jk}^{(-)}L_{ik}^{(+)} = 0 && \hat{k} = \text{odd} \quad i<j \\
&L_{ik}^{(-)}L_{il}^{(+)} - (-1)^{\hat{k}\hat{l}}q^{-1}L_{il}^{(+)}L_{ik}^{(-)} = 0 && \hat{i} = \text{even} \quad k<l \\
&L_{ik}^{(-)}L_{il}^{(+)} - (-1)^{(\hat{k}+1)(\hat{l}+1)}qL_{il}^{(+)}L_{ik}^{(-)} = 0 && \hat{i} = \text{odd} \quad k<l \\
&L_{il}^{(\pm)}L_{jk}^{(\pm)} - (-1)^{(\hat{j}+\hat{k})(\hat{i}+\hat{l})}L_{jk}^{(\pm)}L_{il}^{(\pm)} = 0 && i<j \quad k<l \\
&L_{jl}^{(+)}L_{ik}^{(-)} - (-1)^{(\hat{i}+\hat{k})(\hat{j}+\hat{l})}L_{ik}^{(-)}L_{jl}^{(+)} = 0 && k\leqslant i<j\leqslant l \\
&L_{ik}^{(+)}L_{jl}^{(-)} - (-1)^{(\hat{i}+\hat{k})(\hat{j}+\hat{l})}L_{jl}^{(-)}L_{ik}^{(+)} = 0 && i\leqslant k<l\leqslant j \\
&L_{il}^{(+)}L_{jk}^{(-)} - (-1)^{(\hat{k}+\hat{j})(\hat{i}+\hat{l})}L_{jk}^{(-)}L_{il}^{(+)} \\
&\qquad = (-1)^{\hat{i}\hat{j}+\hat{i}\hat{j}+\hat{i}\hat{l}}(q^{-1}-q)[L_{jl}^{(-)}L_{ik}^{(+)} - L_{jl}^{(+)}L_{ik}^{(-)}] && i<j \quad k<l \\
&(-1)^{\hat{i}(\hat{j}+\hat{l})}L_{jl}^{(\pm)}L_{ik}^{(\pm)} - (-1)^{\hat{k}(\hat{j}+\hat{l})}L_{ik}^{(\pm)}L_{jl}^{(\pm)} \\
&\qquad = (-1)^{\hat{j}\hat{l}}(q^{-1}-q)L_{il}^{(\pm)}L_{jk}^{(\pm)} && i<j \quad k<l.
\end{aligned} \tag{5.7}$$

The co-product Δ, co-unit ε and antipode S maps are defined by the following relations:

$$\Delta(L_{ij}^{(\pm)})=\sum_x L_{ix}^{(\pm)}\otimes L_{xj}^{(\pm)} \qquad \varepsilon(L^{(\pm)}=I \qquad S(L^{(\pm)})L^{(\pm)}=I.$$

We stress that the product rule in verifying these maps is (3.4). It follows from (5.1) that a representation ρ of $\tilde{\mathrm{U}}_q$ is

$$\rho(L_{ij}^{(+)})_\phi^\beta=\tilde{R}_{\phi i}^{\beta j} \qquad \rho(L_{ij}^{(-)})_\phi^\beta=(\tilde{R}^{-1})_{i\phi}^{j\beta}. \tag{5.8}$$

The relations (5.5) are in agreement with the formulas given in [9] where they are discussed in general terms. Here they follow as the existing relations in the space dual to that of the one parameter version of Manin's general linear supergroup; in [9], no relation to Manin's construction is made and only a few special cases of the solutions (1.2) are mentioned. The above construction remains true for general $\alpha(\hat{i},\hat{j})$.

6. Quantized universal enveloping algebra U_q associated to $\overline{\mathrm{GL}}_q((M|N);\alpha)$

U_q is defined as a subalgebra of the dual to A_R; U_q is generated by the unit element $\mathbf{1}'$ and the generators $L_{ij}^{(\pm)}(i,j=1,n)$ which are defined by the following duality relations

$$\begin{aligned}&\langle \mathbf{1}';Z_1Z_2\ldots Z_k\rangle=I^{\otimes k}\\&\langle L^{(\pm)};Z_1Z_2\ldots Z_k\rangle=R_1^{(\pm)}R_2^{(\pm)}\ldots R_k^{(\pm)}\end{aligned} \tag{6.1}$$

where $L^{(\pm)}$, Z_i and $R^{(\pm)}$ are defined in a way identical to that of section 5 with the difference that

$$R^{(+)}=PRP \qquad R^{(-)}=R^{-1} \tag{6.2}$$

where R is defined in (4.6).

Due to the fact that R is a solution of the non-graded Yang–Baxter equation (4.7) and of

$$R_{23}R_{12}^{(-)}R_{13}^{(-)}=R_{13}^{(-)}R_{12}^{(-)}R_{23}. \tag{6.3}$$

It follows that the definition (6.1) is consistent with the relation (4.8); here, the consistency does not depend on particular properties of R. It follows from (6.1) that $L^{(+)}$ and $L^{(-)}$ are upper and lower triangular matrices respectively and that

$$L_{ii}^{(+)}L_{ii}^{(-)}=\mathbf{1}' \qquad (i=1,n) \tag{6.4}$$

with no summation over repeated indices. From the definition (6.1) it can be shown that the following relations exist among the generators

$$RP(L^{(\pm)}\otimes I)P(L^{(\pm)}\otimes I)=(L^{(\pm)}\otimes I)P(L^{(\pm)}\otimes I)PR \tag{6.5a}$$

$$RP(L^{(+)}\otimes I)P(L^{(-)}\otimes I)=(L^{(-)}\otimes I)P(L^{(+)}\otimes I)PR. \tag{6.5b}$$

In verifying the consistency of (6.5) with the definition (6.1) use is made of the fact that R is a solution of (4.7) and of

$$R_{13}R_{23}R_{12}^{(-)}=R_{12}^{(-)}R_{23}R_{13} \tag{6.6a}$$

$$R_{13}R_{23}^{(-)}R_{13}^{(-)}=R_{13}^{(-)}R_{23}^{(-)}R_{12}. \tag{6.6b}$$

This way of defining the quantized universal enveloping algebra is not new and is well described in [10]. Writing (6.5) explicitly we get

$$
\begin{array}{lll}
[L_{ii}^{(+)}, L_{ii}^{(-)}]=0 & \text{for all } i & \\
(L_{ik}^{(\pm)})^2=0 & \hat{\imath}+\hat{k}=\text{odd} & \\
L_{ik}^{(\pm)}L_{jk}^{(\pm)}-(-1)^{\alpha(\hat{\imath},\hat{\jmath})}qL_{jk}^{(\pm)}L_{ik}^{(\pm)}=0 & \hat{k}=\text{even} & i<j \\
L_{ik}^{(\pm)}L_{jk}^{(\pm)}+(-1)^{\alpha(\hat{\imath},\hat{\jmath})}q^{-1}L_{jk}^{(\pm)}L_{ik}^{(\pm)}=0 & \hat{k}=\text{odd} & i<j \\
L_{ik}^{(\pm)}L_{il}^{(\pm)}-(-1)^{\alpha(\hat{k},\hat{l})}qL_{il}^{(\pm)}L_{ik}^{(\pm)}=0 & \hat{\imath}=\text{even} & k<l \\
L_{ik}^{(\pm)}L_{il}^{(\pm)}+(-1)^{\alpha(\hat{k},\hat{l})}q^{-1}L_{il}^{(\pm)}L_{ik}^{(\pm)}=0 & \hat{\imath}=\text{odd} & k<l \\
L_{ik}^{(+)}L_{jk}^{(-)}-(-1)^{\alpha(\hat{\imath},\hat{\jmath})}q^{-1}L_{jk}^{(-)}L_{ik}^{(+)}=0 & \hat{k}=\text{even} & i<j \\
L_{ik}^{(+)}L_{jk}^{(-)}+(-1)^{\alpha(\hat{\imath},\hat{\jmath})}qL_{jk}^{(-)}L_{ik}^{(+)}=0 & \hat{k}=\text{odd} & i<j \\
L_{ik}^{(-)}L_{il}^{(+)}-(-1)^{\alpha(\hat{k},\hat{l})}q^{-1}L_{il}^{(+)}L_{ik}^{(-)}=0 & \hat{\imath}=\text{even} & k<l \\
L_{ik}^{(-)}L_{il}^{(+)}+(-1)^{\alpha(\hat{k},\hat{l})}qL_{il}^{(+)}L_{ik}^{(-)}=0 & \hat{\imath}=\text{odd} & k<l \\
L_{il}^{(\pm)}L_{jk}^{(\pm)}=(-1)^{\alpha(\hat{k},\hat{l})+\alpha(\hat{\imath},\hat{\jmath})}L_{jk}^{(\pm)}L_{il}^{(\pm)} & i<j & k<l \\
L_{jl}^{(+)}L_{ik}^{(-)}=(-1)^{\alpha(\hat{\imath},\hat{\jmath})+\alpha(\hat{k},\hat{l})}L_{ik}^{(-)}L_{jl}^{(+)} & k\leqslant i<j\leqslant l & \\
L_{ik}^{(+)}L_{jl}^{(-)}=(-1)^{\alpha(\hat{\imath},\hat{\jmath})+\alpha(\hat{k},\hat{l})}L_{jl}^{(-)}L_{ik}^{(+)}=0 & i\leqslant k<l\leqslant j &
\end{array}
$$

$$
\begin{aligned}
&(-1)^{\alpha(\hat{\imath},\hat{\jmath})}L_{jl}^{(\pm)}L_{ik}^{(\pm)}-(-1)^{\alpha(\hat{k},\hat{l})}L_{ik}^{(\pm)}L_{jl}^{(\pm)} \\
&\qquad=(q^{-1}-q)L_{il}^{(\pm)}L_{jk}^{(\pm)} \qquad i<j \qquad k<l \\
&(-)^{\alpha(\hat{\imath},\hat{\jmath})}L_{il}^{(+)}L_{jk}^{(-)}-(-1)^{\alpha(\hat{k},\hat{l})}L_{jk}^{(-)}L_{il}^{(+)} \\
&\qquad=(q^{-1}-q)[L_{jl}^{(-)}L_{ik}^{(+)}-L_{jl}^{(+)}L_{ik}^{(-)}] \qquad i<j \qquad k<l.
\end{aligned}
\tag{6.7}
$$

The co-product Δ, co-unit ε and antipode are defined as in section 5; however, the product rule is now (4.4). The fundamental representation ρ is as follows:

$$
\rho(L_{ij}^{(+)})_{\phi}^{\beta}=R_{\phi i}^{\beta j} \qquad \rho(L_{ij}^{(-)})_{\phi}^{\beta}=(R^{-1})_{i\phi}^{j\beta}. \tag{6.8}
$$

In the next two sections we consider in more detail two simple cases.

7. Quantized algebra associated to $GL_q(1|1)$

Although this algebra has already been given in [7, 9] we repeat the exercise for ease of comparison. From the equations (5.5) with format (0, 1) and the identification

$$
L^{+}=\begin{pmatrix} k_1 & (q-q^{-1})X^{+} \\ 0 & k_2 \end{pmatrix} \qquad L^{-}=\begin{pmatrix} k_1^{-} & 0 \\ -(q-q^{-1})X^{-} & k_2^{-} \end{pmatrix} \tag{7.1}
$$

we get

$$
\begin{array}{lll}
(X^{\pm})^2=0 & k_1X^{+}k_1^{-}=qX^{+} & k_2X^{+}k_2^{-}=qX^{+} \\
k_1X^{-}k_1^{-}=q^{-1}X^{-} & k_2X^{-}k_2^{-}=q^{-1}X^{-} & \\
X^{+}X^{-}+X^{-}X^{+}=\dfrac{k_2k_1^{-}-k_1k_2^{-}}{q-q^{-1}} & &
\end{array}
\tag{7.2a}
$$

In terms of the usual Cartan subalgebra generators H_1, H_2 we have that

$$k_1 = q^{H_1/2} \qquad k_2 = q^{H_2/2} \qquad [H_1, X^{\pm}] = \pm 2X^{\pm} \qquad [H_2, X^{\pm}] = \pm 2X^{\pm} \tag{7.2b}$$

The co-product, co-unit ε and antipode S are as follows:

$$\begin{aligned}
&\Delta(k_1^{\pm}) = k_1^{\pm}\otimes k_1^{\pm} \qquad \Delta(k_2^{\pm}) = k_2^{\pm}\otimes k_2^{\pm}\\
&\Delta(X^+) = k_1\otimes X^+ + X^+\otimes k_2 \qquad \Delta(X^-) = X^-\otimes k_1^- + k_2^-\otimes X^-\\
&S(k_1^{\pm}) = k_1^{\mp} \qquad S(k_2^{\pm}) = k_2^{\mp} \qquad S(X^+) = -k_1^- X^+ k_2^- \qquad S(X^-) = -k_2 X^- k_1\\
&\varepsilon(k_1) = \varepsilon(k_2) = 1 \qquad \varepsilon(X^+) = \varepsilon(X^-) = 0.
\end{aligned} \tag{7.2c}$$

Consider the following transformation

$$\chi^+ = -X^+(k_1^- k_2^-)^{1/2} q^{-1/2} \qquad \chi^- = -X^-(k_1^- k_2^-)^{-1/2} q^{-1/2} \qquad K = (k_1^- k_2)^{1/2}.$$

The algebra described in (7.2) can now be written

$$\begin{aligned}
&N \equiv (H_2 - H_1)/2 \qquad H \equiv (H_2 + H_1)/2 \qquad K = q^{N/2}\\
&[H, \chi^{\pm}] = \pm 2\chi^{\pm} \qquad [N, \chi^{\pm}] = 0\\
&(\chi^{\pm})^2 = 0 \qquad \chi^+\chi^- + \chi^-\chi^+ = \frac{q^N - q^{-N}}{q - q^{-1}}\\
&\Delta(\chi^{\pm}) = K^-\otimes \chi^{\pm} + \chi^{\pm}\otimes K\\
&S(K^{\pm}) = K^{\mp} \qquad S(\chi^{\pm}) = -\chi^{\pm} \qquad \varepsilon(\chi^{\pm}) = 0 \qquad \varepsilon(K^{\pm}) = 1.
\end{aligned} \tag{7.3}$$

Denoting the two-dimensional representation by (2), we now consider the decomposition of the tensor product $(2)\otimes(2)$. The parity of $\chi^{\pm}$ and K is $p(\chi^{\pm}) = 1$, $p(K) = 0$. From (5.8) the fundamental representation is

$$\chi^+ = \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix} \qquad \chi^- = \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix} \qquad \text{and} \quad K = \begin{pmatrix} q^{1/2} & 0\\ 0 & q^{1/2}\end{pmatrix}.$$

The states of the representation are

$$|1\rangle \equiv \begin{pmatrix} 0\\ 1\end{pmatrix} \qquad |-1\rangle = \begin{pmatrix} 1\\ 0\end{pmatrix}$$

also $p(|1\rangle) = 0$ and $p(|-1\rangle) = 1$. Using the rule

$$(A\otimes B)|a\rangle\otimes|b\rangle = A|a\rangle\otimes B|b\rangle(-1)^{p(|a\rangle)p(B)}$$

we get $(2)\otimes(2) = (2)^* + (2)^{**}$ with

$$(2)^* = \{|1\rangle\otimes|1\rangle, q^{-1/2}|1\rangle\otimes|-1\rangle + q^{1/2}|-1\rangle\otimes|1\rangle\}$$

and

$$(2)^{**} = \{q^{1/2}|1\rangle\otimes|-1\rangle - q^{-1/2}|-1\rangle\otimes|1\rangle, |-1\rangle\otimes|-1\rangle\}.$$

8. Quantized algebra associated to $\overline{\mathrm{GL}}_q((1|1); \alpha = 0)$

Given the format (0, 1) and the identification

$$L^+ = \begin{pmatrix} k_1 & (q - q^-)X^+\\ 0 & k_2\end{pmatrix} \qquad L^- = \begin{pmatrix} k_1^- & 0\\ (q - q^{-1})X^- & k_2^-\end{pmatrix} \tag{8.1}$$

the equations (6.5) with $\alpha=0$ give the following set of relations

$$
\begin{aligned}
&(X^{\pm})^2=0 \qquad k_1X^{\pm}k_1^{-}=q^{\pm}X^{\pm} \qquad k_2X^{\pm}k_2^{-}=-q^{\pm}X^{\pm}\\
&X^{+}X^{-}-X^{-}X^{+}=\frac{k_2k_1^{-}-k_1k_2^{-}}{q-q^{-1}}.
\end{aligned}
\tag{8.2}
$$

The co-product, co-unit and antipode are identical to the mappings defined in (7.2c). The relations (8.2) are also given in [11]. Consider the following transformations ($i^2=-1$, $\lambda=i$)

$$
\psi^{\pm}\equiv\mp X^{\pm}(k_1^{-}k_2^{-})^{\pm1/2}\left[\frac{(1-\lambda^2)q}{(q-q^{-1})}\right]^{1/2} \qquad K\equiv(k_1^{-}k_2)^{1/2} \tag{8.3}
$$

ψ^{+}, ψ^{-} and K satisfy the following relations

$$
(\psi^{\pm})^2=0 \tag{8.4a}
$$

and

$$
K\psi^{\pm}K^{-}=\lambda^{\pm}\psi^{\pm} \qquad \psi^{+}\psi^{-}-\psi^{-}\psi^{+}=\frac{K^2-K^{-2}}{\lambda-\lambda^{-}} \tag{8.4b}
$$

with co-product, co-unit and antipode defined as follows:

$$
\begin{aligned}
&\Delta(\psi^{\pm})=K^{-}\otimes\psi^{\pm}+\psi^{\pm}\otimes K \qquad \Delta(K^{\pm})=K^{\pm}\otimes K^{\pm}\\
&S(\psi^{\pm})=-\lambda^{\pm}\psi^{\pm} \qquad \varepsilon(K^{\pm})=1 \qquad \varepsilon(\psi^{\pm})=0.
\end{aligned}
\tag{8.4c}
$$

It follows from the first two equations in (8.4b) that

$$
K=\lambda^{H/2} \qquad [H,\psi^{\pm}]=\pm2\psi^{\pm}. \tag{8.4d}
$$

With $q=e^{\eta}$ we have that

$$
\begin{aligned}
&k_1=q^{H_1/2} \qquad [H_1,X^{\pm}]=\pm2X^{\pm}\\
&k_2=k_1K^2=q^{H_2/2} \qquad H_2=H_1+i\pi\eta^{-}H \qquad [H_2,X^{\pm}]=\pm2X^{\pm}(1+i\pi\eta^{-}).
\end{aligned}
\tag{8.5}
$$

Note that following the change of basis described in (8.3), q no longer appears in the relations (8.4); relations (8.4b)-(8.4d) are those of $U_t(\mathrm{sl}(2,\boldsymbol{C}))$ with $t=i$. Denoting the two-dimensional representation by (2), we now consider the decomposition of the tensor product (2)$\otimes$(2). The parities of $\psi^{\pm}$ and K is $p(\psi^{\pm})=1$, and $p(K)=0$. From (6.8) we have

$$
\psi^{+}=\frac{q-q^{-1}}{2}\begin{pmatrix}0&0\\1&0\end{pmatrix} \qquad \psi^{-}=\frac{i(q-q^{-1})}{2}\begin{pmatrix}0&1\\0&0\end{pmatrix}
$$

and

$$
K=q^{1/2}\begin{pmatrix}1&0\\0&i\end{pmatrix}.
$$

The states of the representation are

$$
|1\rangle=\begin{pmatrix}0\\1\end{pmatrix} \qquad \text{and} \qquad |-1\rangle=\begin{pmatrix}1\\0\end{pmatrix}
$$

$p(|1\rangle)=0$ and $p(|-1\rangle)=1$. Using the rule $(A\otimes B)|a\rangle\otimes|b\rangle=A|a\rangle\otimes B|b\rangle$ we get $(2)\otimes(2)=(2)^{*}\oplus(2)^{**}$ with

$$
(2)^{*}=\{|1\rangle\otimes|1\rangle,\ q^{-1/2}|1\rangle\otimes|-1\rangle-q^{1/2}|-1\rangle\otimes|1\rangle\}
$$

and

$$(2)^{**}=\{q^{1/2}|1\rangle\otimes|-1\rangle+q^{-1/2}|-1\rangle\otimes|1\rangle, |-1\rangle\otimes|-1\rangle\}.$$

At this point we want to stress one of the important differences between the two quantized universal enveloping algebras. In the case of $\mathrm{GL}_q(1|1)$ the algebra described in (7.3) has a well known classical limit ($q\to 1$); note that for the two dimensional representation $K(q=1)=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and it follows that the co-product $\Delta(\chi^{\pm})$ is co-commutative. On the other hand, the algebra described in (8.4) is related to $\mathrm{U}_{t=i}(\mathrm{sl}(2, C))$ and therefore it does not have a classical limit; the deformation parameter t has been set equal to i and the parameter q is the free parameter that characterizes the representation (see [11]); in addition, K does not reduce to unity even when $q=1$ and therefore $\Delta(\psi^{\pm})$ is not co-commutative.

9. Concluding remarks

The case examined in section 8 suggests that one of the consequences of choosing the duality condition (4.3) instead of (3.3) is that the algebra obtained does not have a classical limit; the general case remains to be examined. Note that (3.3a) and (4.3a) are both odd pairings. One might consider even pairings with either products (3.4) or (4.4); it is not clear at this time whether this would lead to interesting structures. We had mentioned in the Introduction that the algebra presented in [13] is also related to the solutions (1.2) in the case $q=i$. The connection between the algebra presented in (6.6) and Lee's algebra remains to be established. Finally let us mention that the link polynomials associated with the solutions (1.2) were discussed in [14].

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Note added in proof. It can be shown that the algebra associated to $\mathrm{GL}_q(1|1)$ and $\overline{\mathrm{GL}}_q((1|1); \alpha=0)$ are isomorphic as algebras but differ in their coproducts and antipodes (they have different Hopf structures). See [15] for details.

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